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Lattices generated by strongly closed subgraphs in d-bounded distance-regular graphs

Suogang Gao[a], Jun Guo[b], Wen Liu[a]

[a] *Mathematics and Information College, Hebei Normal University, 050016, China*
[b] *Mathematics and Information College, Langfang Teachers' College, 065000, China*

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Abstract

Let Γ be a d-bounded distance-regular graph with $d \geq 3$. Suppose that $P(x)$ is a set of strongly closed subgraphs containing x and that $P(x, i)$ is a subset of $P(x)$ consisting of the elements of $P(x)$ with diameter i. Let $\mathcal{L}(x, i)$ be the set generated by the intersection of the elements in $P(x, i)$. On ordering $\mathcal{L}(x, i)$ by inclusion or reverse inclusion, $\mathcal{L}(x, i)$ is denoted by $\mathcal{L}_O(x, i)$ or $\mathcal{L}_R(x, i)$. We prove that $\mathcal{L}_O(x, i)$ and $\mathcal{L}_R(x, i)$ are both finite atomic lattices, and give the conditions for them both being geometric lattices. We also give the eigenpolynomial of $P(x)$ on ordering $P(x)$ by inclusion or reverse inclusion.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For a subset $\Delta \subset V(\Gamma)$, we identify Δ with the induced subgraph on Δ and write $\Delta = (V(\Delta), E(\Delta))$.

For two vertices $u, v \in \Gamma$, let $\partial_\Gamma(u, v)$ denote the distance between u and v in Γ, i.e., the length of a shortest path connecting u and v. We also write $\partial(u, v)$ when no confusion occurs. Let

$$d(\Gamma) = \max\{\partial(u, v) \mid u, v \in V(\Gamma)\}$$

and call $d(\Gamma)$ the diameter of Γ. Similarly, the diameter of a subgraph Δ is written as $d(\Delta)$.

E-mail address: sggao@heinfo.net (S. Gao).

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For $u \in V(\Gamma), x \in V(\Delta)$, set

$$\Gamma_i(u) = \{v \in V(\Gamma) \mid \partial_\Gamma(u, v) = i\}, \qquad \Gamma(u) = \Gamma_1(u),$$
$$\Delta_i(x) = \{y \in V(\Delta) | \partial_\Delta(x, y) = i\}, \qquad \Delta(x) = \Delta_1(x).$$

For vertices $u, v \in \Gamma$ with $\partial(u, v) = i$, set

$$C(u, v) = C_i(u, v) = \Gamma_{i-1}(u) \cap \Gamma(v),$$
$$A(u, v) = A_i(u, v) = \Gamma_i(u) \cap \Gamma(v),$$
$$B(u, v) = B_i(u, v) = \Gamma_{i+1}(u) \cap \Gamma(v).$$

For the cardinalities we use lower case letters, i.e.,

$$c_i = c_i(u, v) = |C_i(u, v)|,$$
$$a_i = a_i(u, v) = |A_i(u, v)|,$$
$$b_i = b_i(u, v) = |B_i(u, v)|.$$

A connected graph Γ is said to be distance-regular if c_i, a_i, b_i exist for all $i, 0 \leq i \leq d$, i.e., these numbers depend on only i rather than the individual choice of vertices.

All graphs considered in this paper are distance-regular graphs. The reader is referred to [2,3,5] for general theory of distance-regular graphs.

Recall that a subgraph Δ of Γ is said to be strongly closed if $C(u, v) \cup A(u, v) \subset \Delta$ for every pair of vertices $u, v \in \Delta$ (see [11]). A subspace of Γ is a regular subgraph induced on a strongly closed subset. It is obvious that the strongly closed subgraphs are connected and for all $u, v \in \Delta$, $\partial_\Gamma(u, v) = \partial_\Delta(u, v)$. We use $\langle\langle x, y\rangle\rangle$ to denote the smallest strongly closed subgraph containing x and y for $x, y \in V(\Gamma)$.

Let Γ be a distance-regular graph with diameter d. Γ is said to be d-bounded, if the following (i), (ii) hold.

(i) Every strongly closed subgraph of Γ is regular.
(ii) For all $x, y \in V(\Gamma)$, x and y are contained in a common strongly closed subgraph of diameter $\partial(x, y)$.

It is clear that every strongly closed subgraph in d-bounded distance-regular graphs is a subspace.

Proposition 1.1 (*[14] Lemma 4.2, 4.5*). *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-bounded distance-regular graph with diameter d. Then the following* (i)–(iii) *hold.*

(i) *The intersection of two subspaces is either a subspace or the empty set.*
(ii) *Let Δ be a subspace of Γ. Then Δ is distance-regular with intersection numbers*

$$c_i(\Delta) = c_i, \quad 0 \leq i \leq d(\Delta),$$
$$a_i(\Delta) = a_i, \quad 0 \leq i \leq d(\Delta),$$
$$b_i(\Delta) = b_i - b_{d(\Delta)}, \quad 0 \leq i \leq d(\Delta).$$

(iii) *For any $x, y \in V(\Gamma)$, the subspace of diameter $\partial(x, y)$ containing x, y is unique.*

Proposition 1.2 (*[13] Lemma 2.6*). *Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a d-bounded distance-regular graph with diameter d. Then we have*

$$b_i > b_{i+1}, \quad 0 \leq i \leq d - 1.$$

Now we recall some definitions relevant to lattices. The reader is referred to [1,4,12] for details.

Let P denote a poset with partial order $\leq$, and let $p, q \in P$. As usual, we write $p < q$ whenever $p \leq q$ and $p \neq q$. We say q covers p, denoted by $p < \cdot q$, whenever $p < q$, and there is no $r \in P$ such that $p < r < q$. An element $p \in P$ is said to be minimal (resp. maximal) whenever there is no $q \in P$ such that $q < p$ (resp. $p < q$). Whenever P has a unique minimal (resp. maximal) element, we denote it by 0 (resp. 1), and we say P has a 0 (resp. 1).

Let P be a locally finite poset and R a commutative ring with unit element. Assume $\mu(x, y) : P \longrightarrow R$ is a binary function on the poset P, then $\mu(x, y)$ is called the Möbius function on P if the following (i)–(iii) hold.

(i) For any $x \in P$, $\mu(x, x) = 1$.
(ii) For $x, y \in P$, if $x \leq y$ does not hold, then $\mu(x, y) = 0$.
(iii) For $x, y \in P$, if $x < y$, then $\sum_{x \leq z \leq y} \mu(x, z) = 0$.

By [12, p. 4, Prop. 1.3], the locally finite poset P has a unique Möbius function.

Suppose P is a poset with 0. By an atom in P, we mean an element in P that covers 0. By a rank function on P, we mean a function

$$r : P \to N,$$

such that $r(0) = 0$, and for all $p, q \in P$, if $p < \cdot q$, then we have $r(q) = r(p) + 1$. Here N is the set of nonnegative integers.

Let P be a poset with minimal element 0 and maximal element 1. Assume r is the rank function of P. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{(r(1) - r(a))}$$

is said to be the eigenpolynomial on P.

A poset P is said to be a lattice whenever for any elements $p, q \in P$, the upper bound $a \vee b$ and the lower bound $a \wedge b$ exist.

A lattice P with minimal element 0 is said to be atomic whenever for any $a \in P \backslash \{0\}$, a is an upper bound of some atoms in P. That is $a = \vee \{p \in P | 0 < \cdot p \leq a\}$. It is obvious that if P is finite, P is an atomic lattice if and only if every element in $P \backslash \{0\}$ is an upper bound of finite atoms.

A finite lattice with minimal element 0 is said to be geometric if the following (i), (ii) hold.

(i) Every element in $P \backslash \{0\}$ is an upper bound of finite atoms.
(ii) There exists a rank function r on P such that

$$r(p \wedge q) + r(p \vee q) \leq r(p) + r(q), \tag{1}$$

for any $p, q \in P$.

Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Pick $x \in V(\Gamma)$, and let $P(x)$ be the set of strongly closed subgraphs containing x. For $1 \leq i \leq d - 1$, set

$$P(x, i) = \{\Delta \in P(x) | d(\Delta) = i\}.$$

Suppose $\mathcal{L}(x, i)$ is the set of the intersection of elements in $P(x, i)$ (every element in $P(x, i)$ is the intersection of itself). We make the convention that the intersection of an empty set of elements is Γ. Then $\Gamma \in \mathcal{L}(x, i)$. $\mathcal{L}(x, i)$ is called the set generated by the intersection of elements in $P(x, i)$.

If we define the partial order of $\mathcal{L}(x, i)$ by inclusion (resp. inverse inclusion), i.e., for any $\Delta, \Delta' \in \mathcal{L}(x, i)$,

$$\Delta \leq \Delta' \Longleftrightarrow \Delta \subset \Delta' \quad (\Delta \leq \Delta' \Longleftrightarrow \Delta' \subset \Delta)$$

then $\mathcal{L}(x, i)$ is a poset, which is denoted by $\mathcal{L}_O(x, i)(\mathcal{L}_R(x, i))$.

Let Γ be a d-bounded distance-regular graph with $d \geq 3$. The purpose of this paper is to study the lattices $\mathcal{L}_O(x, i)$ and $\mathcal{L}_R(x, i)$. We prove that $\mathcal{L}_O(x, i)$ and $\mathcal{L}_R(x, i)$ are both finite atomic lattices, and give the conditions for them both being geometric lattices. We also give the eigenpolynomial of $P(x)$ when it has partial order by inclusion (resp. inverse inclusion).

The results on the lattices generated by different transitive sets of subspaces and the geometricity of lattices generated by orbits of subspaces under finite classical groups can be found in Huo et al. [7,8], Huo and Wan [9], Gao and You [6], Orlik and Solomon [10].

2. Preliminary results

Lemma 2.1. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Suppose Δ and Δ' are strongly closed subgraphs with diameter i and $i + s + t$, respectively, and with $\Delta \subset \Delta'$. Then the number of strongly closed subgraphs $\widetilde{\Delta}$ with diameter $i + s$ satisfying $\Delta \subset \widetilde{\Delta} \subset \Delta'$ is determined by i, s and t, independently of the choice of Δ and Δ'; it is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})},$$

where $i + 1 \leq i + s \leq i + s + t \leq d, 0 \leq i \leq d$.

Proof. Let x and y be a pair of vertices in Δ with $\partial(x, y) = i$. By Proposition 1.1(iii) it is obvious that $\Delta = \langle\langle x, y\rangle\rangle$ and $x, y \in \Delta'$. Since $d(\Delta') = i + s + t$, there exists a sequence of vertices $y = u_0, u_1, \ldots, u_s = z$ such that

$$u_{l+1} \in B(x, u_l) \cap \Delta', \quad l = 0, 1, \ldots, s - 1.$$

Also by Proposition 1.1(iii) we know that $\langle\langle x, z\rangle\rangle$, denoted by $\widetilde{\Delta}$, is a strongly closed subgraph of diameter $i + s$ containing Δ. Now we will count the number of $\widetilde{\Delta}$ of this type.

Firstly, by the choice of the sequence of vertices above there are

$$(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})$$

strongly closed subgraphs $\widetilde{\Delta}$ of diameter $i + s$.

Next we will consider the number of times that every $\widetilde{\Delta}$ repeats. It is easy to show that two different sequences of vertices

$$\begin{aligned} &y = u_0, u_1, \ldots, u_s = z, \qquad u_{l+1} \in B(x, u_l) \cap \Delta', \quad l = 0, 1, \ldots, s - 1, \\ &y = u'_0, u'_1, \ldots, u'_s = z', \qquad u'_{l+1} \in B(x, u'_l) \cap \Delta', \quad l = 0, 1, \ldots, s - 1, \end{aligned}$$

determine the same strongly closed subgraph $\widetilde{\Delta}$ if and only if the two sequences lie in the same $\widetilde{\Delta}$. Hence to prove our result it suffices to count the number of sequences of vertices in one strongly closed subgraph $\widetilde{\Delta}$. It is clear that the number of such sequences of vertices is

$$(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s}).$$

So the number of $\widetilde{\Delta}$ of this type is

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}. \quad \square$$

Corollary 2.2. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Then for $1 \leq i \leq d-1$, the following* (i), (ii) *hold.*

(i) *For $i+1 \leq i+t \leq d$, Δ and Δ' are subspaces of Γ with diameter i and $i+t$, respectively, and with $\Delta \subset \Delta'$. Then the number of subspaces $\widetilde{\Delta}$ in Γ with diameter $i+1$ satisfying $\Delta \subset \widetilde{\Delta} \subset \Delta'$ is $(b_i - b_{i+t})/(b_i - b_{i+1})$. In particular, the number of subspaces in Γ with diameter $i+1$ containing Δ is $b_i/(b_i - b_{i+1})$.*

(ii) *For $x \in V(\Gamma)$, the number of subspaces of diameter i in Γ containing x is*

$$\frac{b_0 b_1 \cdots b_{i-1}}{(b_0 - b_i)(b_1 - b_i)\cdots(b_{i-1} - b_i)}.$$

Lemma 2.3. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$ and let $1 \leq i, j \leq d-1$. Then $\mathcal{L}(x,i) \subset \mathcal{L}(x,j)$ if and only if $i \leq j$.*

Proof. If $i = j$, it is easy to see that $\mathcal{L}(x,i) \subset \mathcal{L}(x,j)$. If $i < j$, we first prove $\mathcal{L}(x, j-1) \subset \mathcal{L}(x,j)$ and for this it suffices to prove $P(x, j-1) \subset \mathcal{L}(x,j)$. For any $\Delta \in P(x, j-1)$, we know that there exist two subspaces $\Delta', \Delta'' \in P(x,j)$ such that $\Delta \subset \Delta'$ and Δ'' from Proposition 1.2 and Corollary 2.2(i). Suppose that $\widetilde{\Delta}$ is the intersection of Δ' and Δ'', then it is immediate that $\Delta \subseteq \widetilde{\Delta} \subset \Delta'$ and Δ''. Since $d(\Delta) = j-1$ and $d(\Delta') = d(\Delta'') = j$, $d(\widetilde{\Delta}) = j$ or $j-1$. If $d(\widetilde{\Delta}) = j$, it is clear that $\Delta' = \widetilde{\Delta} = \Delta''$ from Proposition 1.1(iii). This is a contradiction. So $d(\widetilde{\Delta}) = j-1$. Also by Proposition 1.1(iii), $\Delta = \widetilde{\Delta}$. This indicates that Δ is the intersection of Δ' and Δ'', and hence $\Delta \in \mathcal{L}(x,j)$.

Now noting that

$$\mathcal{L}(x,i) \subset \mathcal{L}(x,i+1) \subset \cdots \subset \mathcal{L}(x,j-1) \subset \mathcal{L}(x,j)$$

we have $\mathcal{L}(x,i) \subset \mathcal{L}(x,j)$.

Conversely, observe that $\mathcal{L}(x,i) \subset \mathcal{L}(x,j)$, then $P(x,i) \subset \mathcal{L}(x,j)$. For $\Delta \in P(x,i)$, it is clear that $\Delta \neq \Gamma$. So Δ is the intersection of some elements in $P(x,j)$. Therefore there exists $\Delta' \in P(x,j)$ such that $\Delta \subset \Delta'$. Thus $i \leq j$. □

Lemma 2.4. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$ and let $1 \leq i \leq d-1$. Then $\mathcal{L}(x,i)$ is composed of Γ and the subspaces containing x with diameter $\leq i$ in Γ.*

Proof. Let Δ be a subspace of Γ containing x with $d(\Delta) = j \leq i$. Then there exists $y \in \Delta$ such that $\partial(x,y) = j$. So by Proposition 1.1(iii) we know that $\Delta = \langle\langle x, y\rangle\rangle$. Hence $\Delta \in P(x,j)$. Moreover, by Lemma 2.3 we have $\Delta \in P(x,j) \subset \mathcal{L}(x,j) \subset \mathcal{L}(x,i)$. By the construction of $\mathcal{L}(x,i)$, a subspace except Γ with diameter greater than i is not contained in $\mathcal{L}(x,i)$. The assertion is proved. □

Lemma 2.5. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$ and let $1 \leq i \leq d-1$. Then the following* (i), (ii) *hold.*

(i) *$\mathcal{L}_R(x,i)$ is a lattice with minimal element Γ and maximal element $\{x\}$.*
(ii) *$\mathcal{L}_O(x,i)$ is a lattice with minimal element $\{x\}$ and maximal element Γ.*

Proof. (i) For any $\Delta, \Delta' \in \mathcal{L}_R(x,i)$, since $x \in \Delta$ and $x \in \Delta'$, $\Delta \cap \Delta' \neq \emptyset$. By Proposition 1.1(i) we know that $\Delta \cap \Delta'$ is a subspace. From the definition of $\mathcal{L}_R(x,i)$ we have that Δ and Δ' are both intersections of some elements in $P(x, i)$. Hence $\Delta \vee \Delta'$ is the intersection of some elements in $P(x, i)$. Therefore $\Delta \vee \Delta' \in \mathcal{L}_R(x, i)$. Since $\Gamma \in \mathcal{L}_R(x, i)$, $\Delta \cup \Delta' \subset \Gamma$, and

$$\Delta \wedge \Delta' = \cap\{\widetilde{\Delta} \in \mathcal{L}_R(x, i) \mid \Delta \cup \Delta' \subset \widetilde{\Delta}\},$$

we have that $\Delta \wedge \Delta' \in \mathcal{L}_R(x, i)$. Thus $\mathcal{L}_R(x, i)$ is a lattice.

(ii) Similar to the proof of (i). $\square$

Definition 2.6. Let Γ be a d-bounded distance-regular graph with diameter d. Suppose that Δ_1 and Δ_2 are two subspaces in Γ. Call the smallest subspace containing Δ_1 and Δ_2 the join of Δ_1 and Δ_2, and denote it by $\Delta_1 + \Delta_2$.

Lemma 2.7. *Let Γ be a d-bounded distance-regular graph with diameter d. Suppose Δ and Δ' are the subspaces with diameter $i+s$ and $i+1$, respectively, where $0 \le i \le i+s \le i+s+1 \le d$. If $d(\Delta \cap \Delta') = i$, then*

$$d(\Delta) + d(\Delta') = d(\Delta \cap \Delta') + d(\Delta + \Delta').$$

Proof. Since $d(\Delta \cap \Delta') = i$, there exist $x, y \in \Delta \cap \Delta'$ such that $\partial(x, y) = i$. From $x, y \in \Delta \cap \Delta' \subset \Delta$, we obtain that there exists a sequence of vertices, $x = u_0, u_1, \ldots, u_s = z$, in Δ such that $u_l \in B(y, u_{l-1}) \cap \Delta$, $1 \le l \le s$. Thus $\partial(z, y) = i + s$. It follows from Proposition 1.1(iii) that $\Delta = \langle\langle y, z\rangle\rangle$ and $\Delta \cap \Delta' = \langle\langle y, x\rangle\rangle$. Since $d(\Delta') = i + 1$, there exists $w \in B(x, y) \cap \Delta'$ such that $\partial(x, w) = i + 1$. We conclude that there exists $u \in B(x, y) \cap \Delta'$ such that $\partial(z, u) = i + s + 1$. Indeed, suppose that $\partial(z, u) \le i + s$, for any $u \in B(x, y) \cap \Delta'$. Then $u \in C(z, y) \cup A(z, y) \subset \Delta$, since Δ is a subspace. So $B(x, y) \cap \Delta' \subset \Delta \cap \Delta'$. This contradicts the fact that the diameter of $\Delta \cap \Delta'$ is i. So $\partial(x, u) \ge \partial(z, u) - \partial(z, x) = i + 1$ for the vertex u above. Since $d(\Delta') = i+1$, $\partial(x, u) = i+1$. It follows from Proposition 1.1(iii) that $\Delta' = \langle\langle x, u\rangle\rangle$. Since $\langle\langle z, u\rangle\rangle$ is the smallest subspace with diameter $i + s + 1$ containing both Δ and Δ', $\Delta + \Delta' = \langle\langle z, u\rangle\rangle$. $\square$

Lemma 2.8. *Let Γ be a d-bounded distance-regular graph with diameter d. Suppose Δ and Δ' are the subspaces with diameter $i + s$ and $i + t$, respectively, where $0 \le i \le i + s, i + t \le i + s + t \le d$. If $d(\Delta \cap \Delta') = i$, then $d(\Delta) + d(\Delta') \ge d(\Delta \cap \Delta') + d(\Delta + \Delta')$.*

Proof. We use induction for t. The conclusion is clearly true for $t = 0$. From Lemma 2.7 the conclusion is true for $t = 1$. Suppose that the conclusion is true for $t - 1$. By an argument similar to that of Lemma 2.7, there exist $x, y \in \Delta \cap \Delta'$ and $z \in \Delta$ such that $\partial(x, y) = i$, $\partial(x, z) = s$ and $\partial(y, z) = i + s$. It follows from Proposition 1.1 that $\langle\langle x, y\rangle\rangle = \Delta \cap \Delta'$ and $\langle\langle y, z\rangle\rangle = \Delta$. Since $d(\Delta') = i + t$, there exists a sequence of vertices $v_l \in B(x, v_{l-1}) \cap \Delta'$, where $v_0 = y$, $1 \le l \le t - 1$. Thus $\partial(x, v_{t-1}) = i + t - 1$. Let $\overline{\Delta} = \langle\langle x, v_{t-1}\rangle\rangle$. Then $\overline{\Delta}$ is a subspace with diameter $i + t - 1$ in Δ' containing $\Delta \cap \Delta'$. Write $\Delta + \overline{\Delta} = \Delta^*$. Since $\Delta \cap \overline{\Delta} \subset \Delta \cap \Delta'$ and $\langle\langle x, y\rangle\rangle = \Delta \cap \Delta'$, $\Delta \cap \overline{\Delta} \subset \langle\langle x, y\rangle\rangle$. It follows from Proposition 1.1 that $\Delta \cap \overline{\Delta} = \langle\langle x, y\rangle\rangle$, since $x, y \in \Delta \cap \overline{\Delta}$. So by induction

$$d(\Delta^*) \le d(\Delta) + d(\overline{\Delta}) - d(\Delta \cap \overline{\Delta}) = i + s + t - 1.$$

From the argument above we obtain that $\overline{\Delta} \subset \Delta^* \cap \Delta'$, $d(\Delta^*) \le (i + t - 1) + s$, $d(\overline{\Delta}) = i + t - 1$ and $d(\Delta') = (i + t - 1) + 1$. Since $\overline{\Delta} \subset \Delta^* \cap \Delta' \subset \Delta'$, $\Delta^* \cap \Delta' = \overline{\Delta}$ or Δ'.

If $\Delta^* \cap \Delta' = \Delta'$, then $\Delta' \subset \Delta^*$. Note that Δ^* is the smallest subspace containing both Δ and $\overline{\Delta}$, and $\overline{\Delta} \subset \Delta'$. So Δ^* is the smallest subspace containing both Δ and Δ', that is, $\Delta + \Delta' = \Delta^*$.

Thus

$$d(\Delta + \Delta') = d(\Delta^*) \le i + s + t - 1 < i + s + t = d(\Delta) + d(\Delta') - d(\Delta \cap \Delta').$$

If $\Delta^* \cap \Delta' = \overline{\Delta}$, we write $i' = i + t - 1$. Thus $d(\Delta^*) \le i' + s$, $d(\Delta') = i' + 1$ and $d(\overline{\Delta}) = i'$. It follows from Lemma 2.7 that

$$d(\Delta^* + \Delta') = d(\Delta^*) + d(\Delta') - d(\Delta^* \cap \Delta') \le i + s + t.$$

Since $\Delta + \Delta' \subseteq \Delta^* + \Delta'$,

$$d(\Delta + \Delta') \le i + s + t = d(\Delta) + d(\Delta') - d(\Delta \cap \Delta').$$

From the argument above we have proved that the result is true for t, and for all t by the principle of induction. $\square$

Lemma 2.9. *Let Γ be a d-bounded distance-regular graph with diameter d. Suppose Δ is a subspace with diameter i, where $2 \le i \le d - 2$. Then there exists a subspace Δ' with diameter i in Γ such that $\Delta \cap \Delta' \ne \emptyset$ and $d(\Delta \cap \Delta') = i - 2$.*

Proof. Let $x, y \in \Delta$ with $\partial(x, y) = i$. It follows from Proposition 1.1(iii) that $\Delta = \langle\langle x, y\rangle\rangle$. Take $z \in C(x, y)$ and $w \in C(x, z)$, then $\partial(x, z) = i - 1$ and $\partial(x, w) = i - 2$. Take $u \in B(y, x)$ and $v \in B(y, u)$, then $\partial(u, y) = i + 1$ and $\partial(v, y) = i + 2$. Since $\partial(v, w) \le \partial(v, u) + \partial(u, x) + \partial(x, w) = i$ and $\partial(v, w) \ge \partial(v, y) - \partial(w, y) = i$, $\partial(v, w) = i$. It follows that $\Delta' = \langle\langle v, w\rangle\rangle$ is a subspace with diameter i and $\Delta \cap \Delta' \ne \emptyset$. Note that $\Delta, \Delta' \subset \langle\langle v, y\rangle\rangle$ and $v \in \Delta'$, $y \in \Delta$. It follows from Proposition 1.1(iii) that $\langle\langle v, y\rangle\rangle$ is the smallest subspace containing Δ and Δ'. Thus from Lemma 2.8

$$d(\Delta \cap \Delta') \le d(\Delta) + d(\Delta') - d(\langle\langle v, y\rangle\rangle) = i - 2.$$

Note that $\langle\langle x, w\rangle\rangle \subset \Delta \cap \Delta'$, $\partial(x, w) = i - 2$. It follows from Proposition 1.1(iii) that $\Delta \cap \Delta' = \langle\langle x, w\rangle\rangle$. $\square$

3. Discussions on the geometric property of lattices

Theorem 3.1. *Let Γ be a d-bounded distance-regular graph with diameter $d \ge 3$ and let $1 \le i \le d - 1$. Then the following* (i)–(iv) *hold.*

(i) *$\mathcal{L}_R(x, i)$, $1 \le i \le d - 1$, is a finite atomic lattice.*
(ii) *$\mathcal{L}_R(x, 1)$ is a finite geometric lattice.*
(iii) *$\mathcal{L}_R(x, d - 1)$ is a finite geometric lattice if and only if for any $\Delta_1, \Delta_2 \in P(x)$,*

$$d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) = d(\Delta_1) + d(\Delta_2).$$

(iv) *If $2 \le i \le d - 2$, then $\mathcal{L}_R(x, i)$ is not a finite geometric lattice.*

Proof. (i) By Lemma 2.4 we know that $\mathcal{L}_R(x, i)$ is composed of the subspaces containing x with diameter $\le i$, $1 \le i \le d - 1$, and Γ itself. By Lemma 2.5(i) we have that $\mathcal{L}_R(x, i)$ is a lattice with minimal element Γ. So $P(x, i)$ is the set of atoms in $\mathcal{L}_R(x, i)$. Also by Lemma 2.4 we have that any element of $\mathcal{L}_R(x, i)$ excluding Γ is the upper bound of finite elements in $P(x, i)$. Hence $\mathcal{L}_R(x, i)$ is a finite atomic lattice, $1 \le i \le d - 1$.

(ii) We first define a rank function on $\mathcal{L}_R(x, i)$. For any $\Delta \in \mathcal{L}_R(x, i)$, define

$$r_R(\Delta) = \begin{cases} i + 1 - d(\Delta), & \text{if } \Delta \ne \Gamma, \\ 0, & \text{if } \Delta = \Gamma. \end{cases}$$

It is clear that r_R is a function from $\mathcal{L}_R(x,i)$ to N. We claim that r_R is the rank function on $\mathcal{L}_R(x,i)$. In fact, $r_R(0) = r_R(\Gamma) = 0$. For any $\Delta, \Delta' \in \mathcal{L}_R(x,i)$ with $\Delta' < \cdot \Delta$, if $\Delta' = \Gamma$, then $\Delta \in P(x,i)$ and $r_R(\Delta) = r_R(\Delta') + 1$. If $\Delta' \neq \Gamma$, then $\Delta \subset \Delta'$ and $\Delta \neq \Delta'$. Thus by Proposition 1.1(iii) we know $d(\Delta') - d(\Delta) \geq 1$. In the following we will prove $d(\Delta') - d(\Delta) \leq 1$. Suppose not. Then $d(\Delta') - d(\Delta) > 1$. Let $d(\Delta') = j$ and $d(\Delta) = t$. Then $j - t \geq 2$ and $j \leq i$. Pick $y \in \Delta$ such that $\partial(x,y) = t$. This implies $y \in \Delta'$. So by Proposition 1.1(iii) we have $\Delta = \langle\langle x,y\rangle\rangle$. Pick a sequence of points in Δ', $y = v_0, v_1, \ldots, v_{j-t} = z$, such that

$$v_l \in B(x, v_{l-1}) \cap \Delta', \quad 1 \leq l \leq j - t.$$

Then $\partial(y,z) = j - t$ and $\partial(x,z) = j$. It follows from Proposition 1.1(iii) that $\Delta' = \langle\langle x,z\rangle\rangle$. Set $\widetilde{\Delta} = \langle\langle x, v_{j-t-1}\rangle\rangle$. Then $\partial(x, v_{j-t-1}) = j - 1 > t$, $\Delta \subset \widetilde{\Delta} \subset \Delta'$, $\Delta \neq \widetilde{\Delta}$, and $\Delta' \neq \widetilde{\Delta}$, which is a contradiction with $\Delta' < \cdot \Delta$. From the discussions above we know $d(\Delta') = d(\Delta) + 1$. Hence $r_R(\Delta) = r_R(\Delta') + 1$ and thus r_R is the rank function on $\mathcal{L}_R(x,i)$.

Next, by Lemma 2.4 we know that $\mathcal{L}_R(x,1)$ consists of the subspaces of diameter 1 containing x, $\{x\}$ and Γ. For any $\Delta, \Delta' \in \mathcal{L}_R(x,1)$ with at least one of them being $\{x\}$ or Γ, it is easy to prove $r_R(\Delta \wedge \Delta') + r_R(\Delta \vee \Delta') \leq r_R(\Delta) + r_R(\Delta')$, i.e., (1) holds. In the following we suppose that Δ, Δ' are two different elements in $P(x,1)$. By Proposition 1.1(iii), there exist $y, z \in V(\Gamma)$ such that $\partial(x,y) = \partial(x,z) = 1$, $\Delta = \langle\langle x,y\rangle\rangle$, and $\Delta' = \langle\langle x,z\rangle\rangle$. Now we claim that $\partial(y,z) = 2$. In fact, if $\partial(y,z) = 1$, then, from $\partial(x,y) = \partial(x,z) = 1$, we have that $z \in \langle\langle x,y\rangle\rangle$. So $\langle\langle x,z\rangle\rangle \subset \langle\langle x,y\rangle\rangle$. Also by Proposition 1.1(iii) we know that $\langle\langle x,z\rangle\rangle = \langle\langle x,y\rangle\rangle$. This is a contradiction. Hence by Lemma 2.8 we conclude that

$$d(\Delta \cap \Delta') \leq d(\Delta) + d(\Delta') - d(\langle\langle y,z\rangle\rangle) = 0,$$

which shows $\Delta \vee \Delta' = \{x\}$. It follows from $\Delta \wedge \Delta' = \Gamma$ that

$$r_R(\Delta \wedge \Delta') + r_R(\Delta \vee \Delta') = 2 = r_R(\Delta) + r_R(\Delta').$$

Hence from (1) we have that $\mathcal{L}_R(x,1)$ is a finite geometric lattice.

(iii) We first note that $\mathcal{L}_R(x, d-1) = P(x)$. This implies that for any $\Delta_1, \Delta_2 \in P(x)$, $\Delta_1 \cap \Delta_2 = \Delta_1 \vee \Delta_2$ and $\Delta_1 + \Delta_2 = \Delta_1 \wedge \Delta_2$.

Suppose for any $\Delta_1, \Delta_2 \in P(x)$, $d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) = d(\Delta_1) + d(\Delta_2)$. Then

$$\begin{aligned} r_R(\Delta_1 \vee \Delta_2) + r_R(\Delta_1 \wedge \Delta_2) &= d - d(\Delta_1 \cap \Delta_2) + d - d(\Delta_1 + \Delta_2) \\ &= d - d(\Delta_1) + d - d(\Delta_2) \\ &= r_R(\Delta_1) + r_R(\Delta_2). \end{aligned}$$

This shows that $\mathcal{L}_R(x, d-1)$ is a finite geometric lattice.

Conversely, let $\mathcal{L}_R(x, d-1)$ be a finite geometric lattice. Then for any $\Delta_1, \Delta_2 \in P(x)$,

$$r_R(\Delta_1 \vee \Delta_2) + r_R(\Delta_1 \wedge \Delta_2) \leq r_R(\Delta_1) + r_R(\Delta_2).$$

Namely,

$$d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) \geq d(\Delta_1) + d(\Delta_2).$$

It follows from Lemma 2.8 that

$$d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2) = d(\Delta_1) + d(\Delta_2).$$

(iv) In the case of $2 \leq i \leq d-2$, let $\partial(x, y) = i$ and $\Delta = \langle\langle x, y\rangle\rangle$. By Lemma 2.9, there exists a subspace Δ' with diameter i in $\mathcal{L}_R(x, i)$ such that $d(\Delta \cap \Delta') = i - 2$. So we have $\Delta \wedge \Delta' = \Gamma$ from Lemma 2.8 and Proposition 1.1. This shows, with the notation $r_R(\Delta)$ from (ii), that

$$r_R(\Delta \wedge \Delta') + r_R(\Delta \vee \Delta') = 3 > 2 = r_R(\Delta) + r_R(\Delta').$$

From (1) we know that $\mathcal{L}_R(x, i)$ is not a finite geometric lattice. $\square$

Theorem 3.2. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$ and let $1 \leq i \leq d-1$. Then $\mathcal{L}_O(x, i)$ is a finite geometric lattice.*

Proof. By Lemma 2.4 we know that $\mathcal{L}_R(x, i)$ is composed of the subspaces with diameter $\leq i$ in Γ and Γ itself and that $\mathcal{L}_O(x, i)$ is a lattice with minimal element $\{x\}$ from Lemma 2.5(ii). So $P(x, 1)$ is a set of atoms with $P(x, 1) \subset \mathcal{L}_O(x, i)$. Next we will prove that any element in $\mathcal{L}_O(x, i)$ excluding $\{x\}$ can be expressed as an upper bound of some elements in $P(x, 1)$. Since $\mathcal{L}_O(x, i) \subset P(x)$, it suffices to prove that every element of $P(x)\backslash\{x\} = \bigcup_{1 \leq j \leq d} P(x, j)$ has such properties. Now we show that any element of $P(x, j)$, $1 \leq j \leq d$, can be expressed as an upper bound of some elements of $P(x, 1)$ by induction. The result is true for $j = 1$. Suppose the result is true for $j = k$. Then for any $\Delta \in P(x, k+1)$, from Proposition 1.1(ii) and Corollary 2.2(ii), we have that the number of subspaces with diameter k in Δ containing x is

$$\varepsilon = \frac{(b_0 - b_{k+1})(b_1 - b_{k+1}) \cdots (b_{k-1} - b_{k+1})}{(b_0 - b_k)(b_1 - b_k) \cdots (b_{k-1} - b_k)}.$$

It is immediate that $\varepsilon \geq 2$ from Proposition 1.2. Therefore there exist two subspaces $\Delta', \Delta'' \in P(x, k)$ such that $\Delta', \Delta'' \subset \Delta$. Let $\widetilde{\Delta}$ be the upper bound of Δ' and Δ''. Then $\Delta', \Delta'' \subset \widetilde{\Delta} \subset \Delta$. Thus $d(\widetilde{\Delta}) = k$ or $k + 1$. If $d(\widetilde{\Delta}) = k$, then $\Delta' = \widetilde{\Delta} = \Delta''$ from Proposition 1.1(iii) and this is a contradiction. Hence $d(\widetilde{\Delta}) = k + 1$. We have $\Delta = \widetilde{\Delta}$ also by Proposition 1.1(iii). This shows that Δ can be expressed as an upper bound of some elements in $P(x, k)$. By induction Δ is an upper bound of some elements in $P(x, 1)$. Therefore $\mathcal{L}_O(x, i)$ is a finite atomic lattice.

For any $\Delta \in \mathcal{L}_O(x, i)$, we define

$$r_O(\Delta) = \begin{cases} d(\Delta), & \text{if } \Delta \neq \Gamma, \\ i + 1, & \text{if } \Delta = \Gamma. \end{cases}$$

By using the same method as in Theorem 3.1 we can prove that r_O is the rank function on $\mathcal{L}_O(x, i)$.

For any $\Delta, \Delta' \in \mathcal{L}_O(x, i)$, if $d(\Delta \vee \Delta') > i$, then $\Delta \vee \Delta' = \Gamma$. By Lemma 2.8 we have

$$r_O(\Delta \wedge \Delta') + r_O(\Delta \vee \Delta') = d(\Delta \wedge \Delta') + i + 1 \leq d(\Delta) + d(\Delta') = r_O(\Delta) + r_O(\Delta').$$

If $d(\Delta \vee \Delta') \leq i$, also by Lemma 2.8 we have

$$\begin{aligned} r_O(\Delta \wedge \Delta') + r_O(\Delta \vee \Delta') &= d(\Delta \wedge \Delta') + d(\Delta \vee \Delta') \leq d(\Delta) + d(\Delta') \\ &= r_O(\Delta) + r_O(\Delta'). \end{aligned}$$

Thus from (1) we know that $\mathcal{L}_O(x, i)$ is a finite geometric lattice. $\square$

4. Calculation of the eigenpolynomial

If the partial order on $P(x)$ is defined by inclusion (resp. inverse conclusion), $P(x)$ is denoted by $P_O(x)$ (resp. $P_R(x)$).

Theorem 4.1. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Then the Möbius function of $P_O(x)$ is*

$$\mu(\Delta, \Delta') = \begin{cases} (-1)^s \dfrac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})}, & \text{if } \Delta \leq \Delta', \\ 0, & \text{otherwise}, \end{cases}$$

where $d(\Delta) = i$ and $d(\Delta') = i + s$. In particular, we put $\mu(\Delta, \Delta') = 1$ if $s = 0$ and $\mu(\Delta, \Delta') = -1$ if $s = 1$.

Proof. For any $\Delta \in P_O(x)$ it is clear that $\mu(\Delta, \Delta) = 1$. For $\Delta, \Delta' \in P_O(x)$ with $\Delta < \Delta'$, $d(\Delta) = i$ and $d(\Delta') = i + s$, by Lemma 2.1, we have

$$\begin{aligned} \sum_{\Delta \leq \widetilde{\Delta} \leq \Delta'} \mu(\Delta, \widetilde{\Delta}) &= 1 + (-1)^1 \frac{b_i - b_{i+s}}{b_i - b_{i+1}} + (-1)^2 \frac{b_{i+1} - b_{i+2}}{b_i - b_{i+1}} \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s})}{(b_i - b_{i+2})(b_{i+1} - b_{i+2})} \\ &\quad + \cdots + (-1)^{s-1} \frac{(b_{i+1} - b_{i+s-1}) \cdots (b_{i+s-2} - b_{i+s-1})}{(b_i - b_{i+1}) \cdots (b_i - b_{i+s-2})} \\ &\quad \times \frac{(b_i - b_{i+s}) \cdots (b_{i+s-2} - b_{i+s})}{(b_i - b_{i+s-1}) \cdots (b_{i+s-2} - b_{i+s-1})} \\ &\quad + (-1)^s \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})} \\ &= 1 + (-1)^1 \frac{b_i - b_{i+s}}{b_i - b_{i+1}} + (-1)^2 \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2})} \\ &\quad + \cdots + (-1)^{s-1} \frac{(b_i - b_{i+s}) \cdots (b_{i+s-2} - b_{i+s})}{(b_i - b_{i+1}) \cdots (b_i - b_{i+s-1})} \\ &\quad + (-1)^s \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})}. \end{aligned}$$

We claim that the sum from the first term to the k-th term on the right-hand side of the equation $(2 \leq k \leq s)$ is

$$S_k = (-1)^{k-1} \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+k-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+k-1})}. \tag{2}$$

Indeed, in the case of $k = 2$,

$$S_2 = 1 + (-1)^1 \frac{b_i - b_{i+s}}{b_i - b_{i+1}} = (-1)^1 \frac{b_{i+1} - b_{i+s}}{b_i - b_{i+1}}.$$

Suppose the result is true for $k \leq s - 1$. Then by induction

$$\begin{aligned} S_{k+1} &= S_k + (-1)^k \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+k-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+k})} \\ &= (-1)^{k-1} \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+k-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+k-1})} \\ &\quad + (-1)^k \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+k-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+k})} \\ &= (-1)^k \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+k} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+k})}. \end{aligned}$$

Hence,

$$\sum_{\Delta \leq \widetilde{\Delta} \leq \Delta'} \mu(\Delta, \widetilde{\Delta}) = S_s + (-1)^s \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})} = 0.$$

□

Theorem 4.2. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Then the eigenpolynomial of $P_O(x)$ is*

$$\chi(P_O(x), t) = \sum_{s=0}^{d} (-1)^s \frac{b_0 b_1 \cdots b_{s-1}}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_s)} t^{d-s}.$$

Proof. From the proof of Theorem 3.2 we know that for any $\Delta \in P_O(x)$, $r(\Delta) = d(\Delta)$ is a rank function on $P_O(x)$. Hence,

$$\chi(P_O(x), t) = \sum_{\Delta \in P_O(x)} \mu(\{x\}, \Delta) t^{r(\Gamma) - r(\Delta)}.$$

For $\Delta, \Delta' \in P_O(x)$ with $d(\Delta) = d(\Delta')$, we have

$$t^{r(\Gamma)-r(\Delta)} = t^{d-d(\Delta)} = t^{d-d(\Delta')} = t^{r(\Gamma)-r(\Delta')}.$$

It follows from Lemma 2.1 and Theorem 4.1 that

$$\begin{aligned}
\chi(P_O(x), t) &= t^d + (-1)^1 \frac{b_0}{b_0 - b_1} t^{d-1} + (-1)^2 \frac{b_1 - b_2}{b_0 - b_1} \frac{b_0 b_1}{(b_0 - b_2)(b_1 - b_2)} t^{d-2} \\
&\quad + \cdots + (-1)^i \frac{(b_1 - b_i)(b_2 - b_i) \cdots (b_{i-1} - b_i)}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_{i-1})} \\
&\quad \times \frac{b_0 b_1 \cdots b_{i-1}}{(b_0 - b_i)(b_1 - b_i) \cdots (b_{i-1} - b_i)} t^{d-i} \\
&\quad + \cdots + (-1)^{d-1} \frac{(b_1 - b_{d-1})(b_2 - b_{d-1}) \cdots (b_{d-2} - b_{d-1})}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_{d-2})} \\
&\quad \times \frac{b_0 b_1 \cdots b_{d-2}}{(b_0 - b_{d-1})(b_1 - b_{d-1}) \cdots (b_{d-2} - b_{d-1})} t \\
&\quad + (-1)^d \frac{b_0 b_1 \cdots b_{d-1}}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_{d-1})} \\
&= \sum_{s=0}^{d} (-1)^s \frac{b_0 b_1 \cdots b_{s-1}}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_s)} t^{d-s}. \quad \square
\end{aligned}$$

Lemma 4.3. *Let $a_1, a_2, \ldots, a_n$ be n distinct numbers. Then*

$$\sum_{j=1}^{n} \frac{1}{\prod\limits_{1 \leq i \leq n, i \neq j} (a_i - a_j)} = 0.$$

Proof. We use induction for n. It is obvious that the lemma holds in the case of $n = 2$. Suppose the conclusion is true for $n = k$. Then for $n = k + 1$, we have

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{\prod\limits_{1\le i\le k+1, i\ne j}(a_i - a_j)} &= \frac{1}{\prod\limits_{i=2}^{k+1}(a_i - a_1)} + \sum_{j=2}^{k+1} \frac{1}{\prod\limits_{1\le i\le k+1, i\ne j}(a_i - a_j)} \\
&= \frac{1}{a_{k+1} - a_1}\left(-\sum_{j=2}^{k} \frac{1}{\prod\limits_{1\le i\le k, i\ne j}(a_i - a_j)}\right) \\
&\quad + \sum_{j=2}^{k+1} \frac{1}{\prod\limits_{1\le i\le k+1, i\ne j}(a_i - a_j)} \\
&= \frac{-1}{a_{k+1} - a_1}\left(\sum_{j=2}^{k+1} \frac{1}{\prod\limits_{2\le i\le k+1, i\ne j}(a_i - a_j)}\right) \\
&= 0. \quad \square
\end{aligned}$$

Theorem 4.4. *Let Γ be a d-bounded distance-regular graph with diameter $d \ge 3$. Then the Möbius function on $P_R(x)$ is*

$$\mu(\Delta, \Delta') = \begin{cases} (-1)^s \dfrac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})}, & \text{if } \Delta \le \Delta', \\ 0, & \text{otherwise}, \end{cases}$$

where $d(\Delta) = i + s$ and $d(\Delta') = i$. In particular, we set $\mu(\Delta, \Delta') = 1$ if $s = 0$ and $\mu(\Delta, \Delta') = -1$ if $s = 1$.

Proof. For any $\Delta \in P_R(x)$, it is obvious that $\mu(\Delta, \Delta) = 1$. For $\Delta, \Delta' \in P_R(x)$ with $\Delta < \Delta'$, $d(\Delta) = i + s$ and $d(\Delta') = i$, we have by Lemma 2.1

$$\begin{aligned}
\sum_{\Delta \le \widetilde{\Delta} \le \Delta'} \mu(\Delta, \widetilde{\Delta}) &= 1 + (-1)^1 \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-2} - b_{i+s})}{(b_i - b_{i+s-1})(b_{i+1} - b_{i+s-1}) \cdots (b_{i+s-2} - b_{i+s-1})} \\
&\quad + (-1)^2 \frac{b_{i+s-1} - b_{i+s}}{b_{i+s-2} - b_{i+s-1}} \\
&\quad \times \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-3} - b_{i+s})}{(b_i - b_{i+s-2})(b_{i+1} - b_{i+s-2}) \cdots (b_{i+s-3} - b_{i+s-2})} \\
&\quad + (-1)^3 \frac{(b_{i+s-2} - b_{i+s})(b_{i+s-1} - b_{i+s})}{(b_{i+s-3} - b_{i+s-2})(b_{i+s-3} - b_{i+s-1})} \\
&\quad \times \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-4} - b_{i+s})}{(b_i - b_{i+s-3})(b_{i+1} - b_{i+s-3}) \cdots (b_{i+s-4} - b_{i+s-3})} \\
&\quad + \cdots + (-1)^{s-1} \frac{(b_{i+2} - b_{i+s})(b_{i+3} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_{i+1} - b_{i+2})(b_{i+1} - b_{i+3}) \cdots (b_{i+1} - b_{i+s-1})} \\
&\quad \times \frac{b_i - b_{i+s}}{b_i - b_{i+1}} \\
&\quad + (-1)^s \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})}.
\end{aligned}$$

Denote by S the right-hand side of the equality. It suffices to prove that $S = 0$. It is easy to see that $S = 0$ if and only if

$$\begin{aligned}
&1 + (-1)^2 \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-2} - b_{i+s})(b_{i+s-1} - b_{i+s})}{(b_i - b_{i+s-1})(b_{i+1} - b_{i+s-1}) \cdots (b_{i+s-2} - b_{i+s-1})(b_{i+s} - b_{i+s-1})} \\
&\quad + (-1)^3 \frac{b_{i+s-1} - b_{i+s}}{b_{i+s-2} - b_{i+s-1}} \\
&\quad \times \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-3} - b_{i+s})(b_{i+s-2} - b_{i+s})}{(b_i - b_{i+s-2})(b_{i+1} - b_{i+s-2}) \cdots (b_{i+s-3} - b_{i+s-2})(b_{i+s} - b_{i+s-2})} \\
&\quad + (-1)^4 \frac{(b_{i+s-2} - b_{i+s})(b_{i+s-1} - b_{i+s})}{(b_{i+s-3} - b_{i+s-2})(b_{i+s-3} - b_{i+s-1})} \\
&\quad \times \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-4} - b_{i+s})(b_{i+s-3} - b_{i+s})}{(b_i - b_{i+s-3})(b_{i+1} - b_{i+s-3}) \cdots (b_{i+s-4} - b_{i+s-3})(b_{i+s} - b_{i+s-3})} \\
&\quad + \cdots + (-1)^s \frac{(b_{i+2} - b_{i+s})(b_{i+3} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{(b_{i+1} - b_{i+2})(b_{i+1} - b_{i+3}) \cdots (b_{i+1} - b_{i+s-1})} \\
&\quad \times \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s})}{(b_i - b_{i+1})(b_{i+s} - b_{i+1})} \\
&\quad + (-1)^{s+1} \frac{(b_{i+1} - b_{i+s})(b_{i+2} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})(b_i - b_{i+s})}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s-1})(b_{i+s} - b_i)} = 0.
\end{aligned}$$

Namely,

$$-1 = \sum_{t=0}^{s-1} \frac{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}{\prod\limits_{0 \le j \le s, j \ne t} (b_{i+j} - b_{i+t})}.$$

Transposing and canceling the numerators,

$$\sum_{t=0}^{s} \frac{1}{\prod\limits_{0 \le j \le s, j \ne t} (b_{i+j} - b_{i+t})} = 0.$$

The assertion is proved by Lemma 4.3. $\square$

Theorem 4.5. *Let Γ be a d-bounded distance-regular graph with $d \geq 3$. Then the eigenpolynomial of $P_R(x)$ is*

$$\chi(P_R(x), t) = t^d - \sum_{i=0}^{d-1} \frac{b_0 b_1 \cdots b_{i-1} b_{i+1} \cdots b_{d-1}}{(b_0 - b_i)(b_1 - b_i) \cdots (b_{i-1} - b_i)(b_{i+1} - b_i) \cdots (b_{d-1} - b_i)} t^i.$$

Proof. It follows from the proof of Theorem 3.1 that for any $\Delta \in P_R(x)$, $r(\Delta) = d - d(\Delta)$ is the rank function on $P_R(x)$. So,

$$\chi(P_R(x), t) = \sum_{\Delta \in P_R(x)} \mu(\Gamma, \Delta) t^{r(\{x\}) - r(\Delta)}.$$

For $\Delta, \Delta' \in P_R(x)$ with $d(\Delta) = d(\Delta')$, we have

$$t^{r(\{x\}) - r(\Delta)} = t^{d(\Delta)} = t^{d(\Delta')} = t^{r(\{x\}) - r(\Delta')}.$$

It follows from Lemma 2.1 and Theorem 4.4 that

$$\begin{aligned}\chi(P_R(x),t) &= t^d + (-1)^1 \frac{b_0 b_1 \cdots b_{d-2}}{(b_0 - b_{d-1})(b_1 - b_{d-1}) \cdots (b_{d-2} - b_{d-1})} t^{d-1} \\ &\quad + (-1)^2 \frac{b_{d-1}}{b_{d-2} - b_{d-1}} \frac{b_0 b_1 \cdots b_{d-3}}{(b_0 - b_{d-2})(b_1 - b_{d-2}) \cdots (b_{d-3} - b_{d-2})} t^{d-2} \\ &\quad + \cdots + (-1)^{d-i} \frac{b_{i+1} b_{i+2} \cdots b_{d-1}}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{d-1})} \\ &\quad \times \frac{b_0 b_1 \cdots b_{i-1}}{(b_0 - b_i)(b_1 - b_i) \cdots (b_{i-1} - b_i)} t^i \\ &\quad + \cdots + (-1)^{d-1} \frac{b_2 b_3 \cdots b_{d-1}}{(b_1 - b_2)(b_1 - b_3) \cdots (b_1 - b_{d-1})} \frac{b_0}{b_0 - b_1} t \\ &\quad + (-1)^d \frac{b_1 b_2 \cdots b_{d-1}}{(b_0 - b_1)(b_0 - b_2) \cdots (b_0 - b_{d-1})} \\ &= t^d - \sum_{i=0}^{d-1} \frac{b_0 b_1 \cdots b_{i-1} b_{i+1} \cdots b_{d-1}}{(b_0 - b_i)(b_1 - b_i) \cdots (b_{i-1} - b_i)(b_{i+1} - b_i) \cdots (b_{d-1} - b_i)} t^i.\end{aligned}$$

□

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